Lecture 7

Plan:

✓ 0. Discuss pset

✓ 1. Finish LP duality (prev notes).

2. Faces of Polyhedra

Faces of Polyhedra

Def: $a^{(1)}, \dots, a^{(k)} \in \mathbb{R}^n$ are

affinely independent if

$$\sum_{i=1}^{k} \lambda_i a^{(i)} = 0$$

and $\sum \lambda_i = 0$ imply $\lambda_1 = \dots = \lambda_k = 0$.

(w/out $\sum \lambda_i = 0$, is just linear indp.)

linear independence $\Rightarrow$ affine independ.

Note:

$\{a^{(i)}\}$ affinely independent iff

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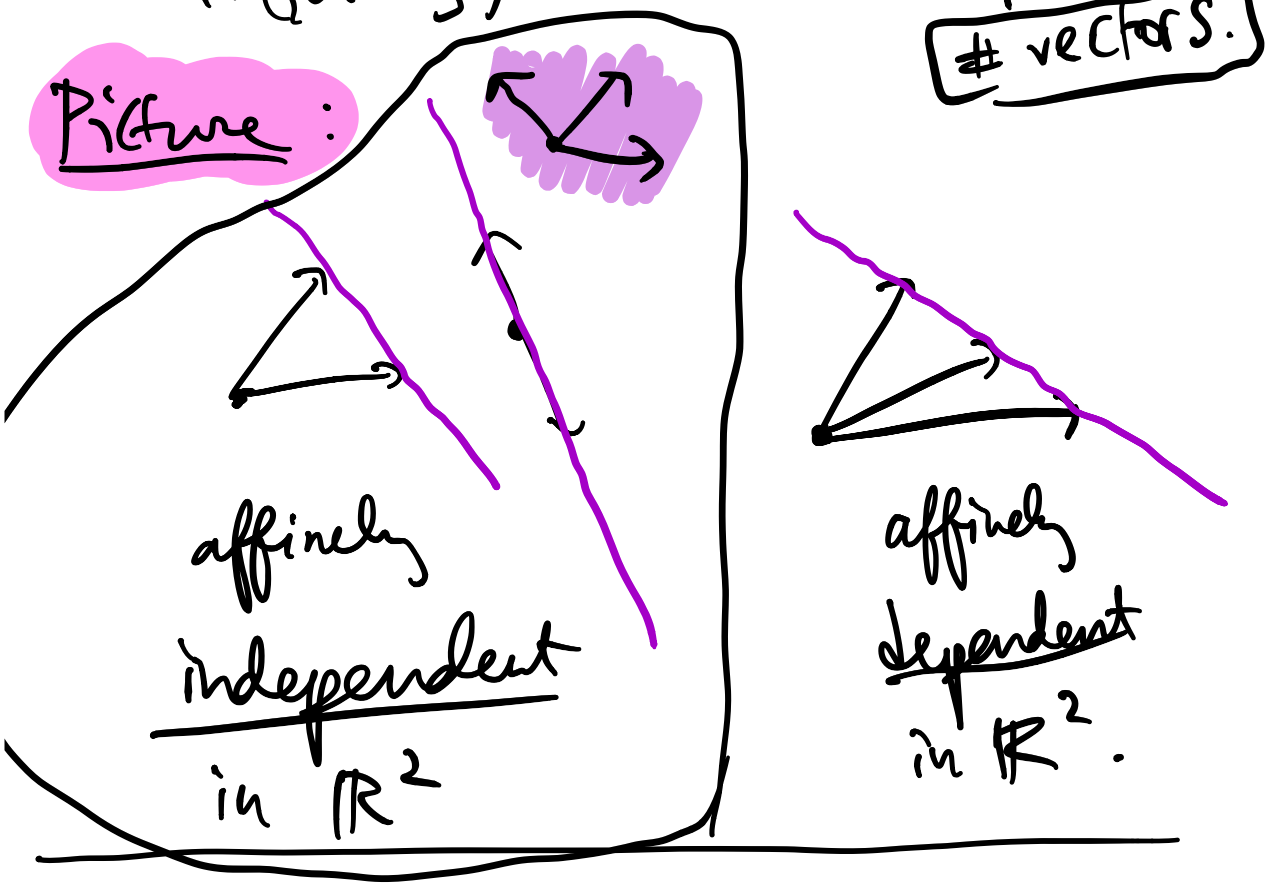
$\left\{ \begin{bmatrix} a^{(i)} \\ 1 \end{bmatrix} \right\}$ <u>linearly</u> independent.

$\Leftrightarrow$ $\mathrm{aff}(\{a^{(i)}\})$ has dimension $k-1$

k = $\boxed{\#\text{ vectors.}}$

<u>Picture</u>:



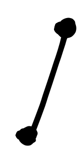
<u>Def</u> <u>Dimension</u> $\dim(P)$ of polyhedron P:

$-1 +$ max # affinely independent points in P.

Equivalently, dimension of affine hull $\mathrm{aff}(P)$.

Examples: $P = \emptyset$, $\dim(P) = -1$

$P =$ singleton  $\dim(P) = 0$

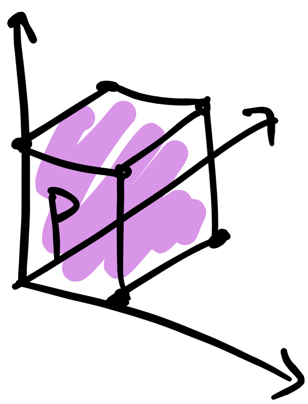
$P =$ line segment  $\dim(P) = 1$

$\vdots$

$\mathrm{aff}(P) = \mathbb{R}^n$ $\qquad$ $\dim(P) = n$;

P "full dimensional"

e.g. cube in $\mathbb{R}^3$: $\{x : 0 \le x_i \le 1\}$



$\dim P = 3$

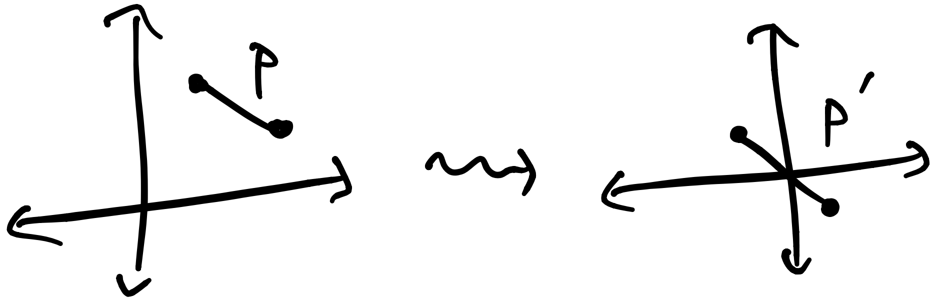
$\dim \mathbb{R}^3 = 3$

(as polyhedron).

Why <u>affine</u>, not <u>linear</u>? affine independence is translation invariant:

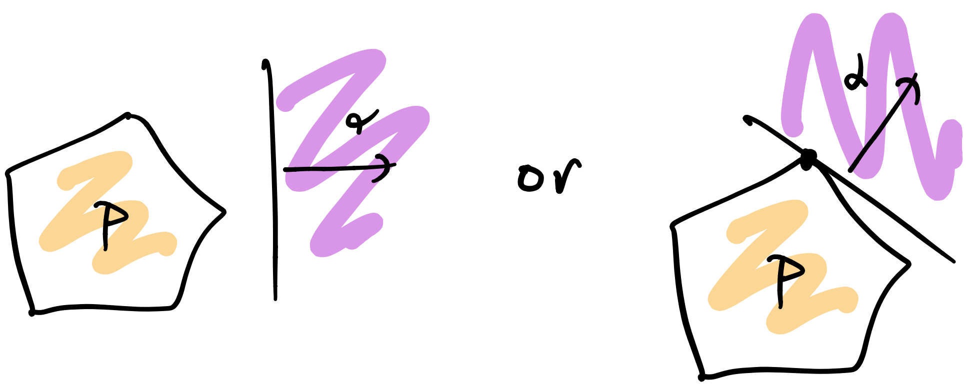
if I used max # lin indep points -1

$\dim(P) = 1$ $\qquad$ $\dim(P') = 0$



$1 = \dim(P) = \dim(P')$.

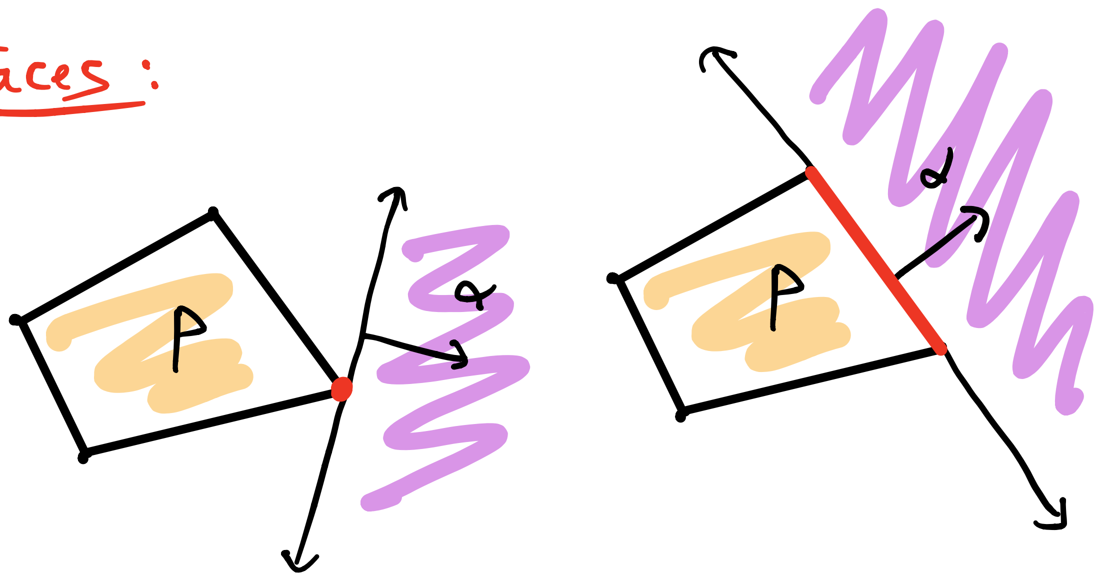
Def: $a^T x \le \beta$ is a <u>valid inequality</u> for P if $a^T x \le \beta$ for all $x \in P$.



Def A <u>face</u> of a polyhedron P is $\{x \in P : \quad \}$ for

$\alpha^T x \le \beta$ valid.

Faces:



Properties:

- Faces are polyhedra
- Empty face & entire P are called trivial faces
- else F nontrivial

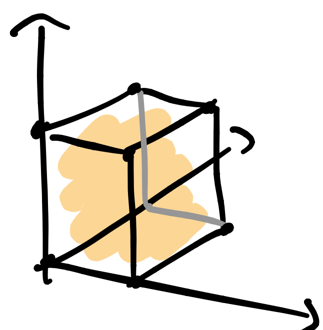
$\le \dim(F) \le$

- $F : \dim(F) = \dim(P) - 1$ called facets.
- $F : \dim(F) = 0$ called vertices

- F : $\dim(F)=0$ called **vertices**

<u>Ex</u>: list the 28 faces of the cube

$$P = \{x \in \mathbb{R}^3 : \qquad \}$$

$P =$ 

<u>Fact</u>: ∞ many valid ineqs, but # faces finite!

<u>Theorem</u>: ("Faces" theorem)

Let $A \in \mathbb{R}^{m \times n}$, $\quad A = \begin{bmatrix} \vdots \\ - \; a_i^T \; - \\ \vdots \end{bmatrix}$

Any nonempty face of $P = \{x : Ax \le b\}$

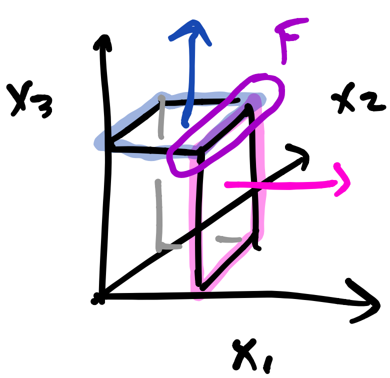
is

$$\left\{ x : \qquad \qquad \qquad \qquad \right\}$$

for some set $I \subseteq \{1, \dots, m\}$.

$\Rightarrow$

E.g. cube



$$F = \left\{ x : \qquad \qquad \qquad \right\}$$

Proof Consider valid inequality $a^T x \leq b$ giving nonempty face F.

- $F =$ optimum solutions to bounded LP

$$\boxed{\begin{array}{ll} & \text{max} \\ (P) & \text{subject to} \end{array}}$$

- Let y^* optimal solution to dual.

- Complementary slackness:

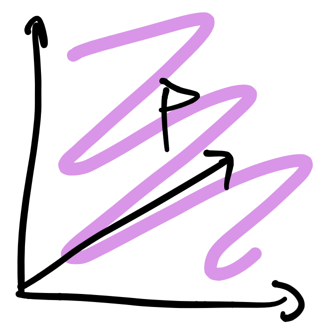
  optimal solns F are

S_v : $\}$.

Thus we can take $I = \{i : y_i^* > 0\}$. $\square$

Ex: positive orthant $\{x \in \mathbb{R}^n : x_i \geq 0\} = P$ has $2^n + 1$ faces

- How many of dim k?



For poly<u>topes</u> can bound # faces in terms of # vertices.

"upper bound theorem"

Dehn-Sommerville equation

For extreme points (dimension 0 faces), can just use <u>equalities</u>.

Theorem ("Vertex" Theorem)

Let x^* extreme point for

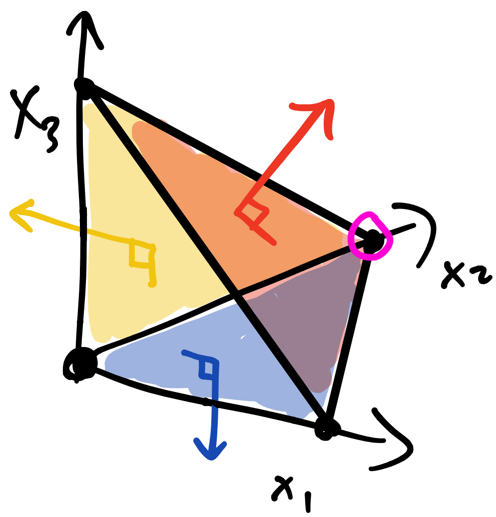
$$A = \begin{bmatrix} \vdots \\ - \; a_i^T \; - \\ \vdots \end{bmatrix}$$

$$P = \{x : Ax \leq b\}.$$

Then $\exists\, I$ s.t. x^* is the unique soln to

moreover, any such unique solution x^* is extreme.

e.g. simplex $(0,1,0)$ is intersection of 3 constraints



Proof: Given extreme point x^*,

- define $I = \{ i : \quad \}$.
- Note for $i \notin I$, $\boxed{}$.
- By "faces theorem", x^* uniquely defined by

(*) $\qquad i \in I$

(**) $\qquad i \notin I.$

- Suppose $\exists$ other soln. $\hat{x}$ to (*).
- Because $\qquad$ for $i \notin I$,

still satisfies (*), (* *) for

- Contradicts F having only one point. $\square$.

Basic Feasible Solutions:

For $\qquad P = \{ \quad , \quad \}$

can describe extreme points very explicitly.

($\qquad$).

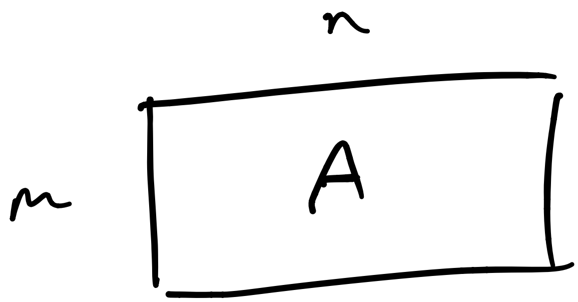
Corollary of Vertex Thm: Extreme pts. of $P = \{ \quad \}$ come from setting

$\boxed{\quad}$

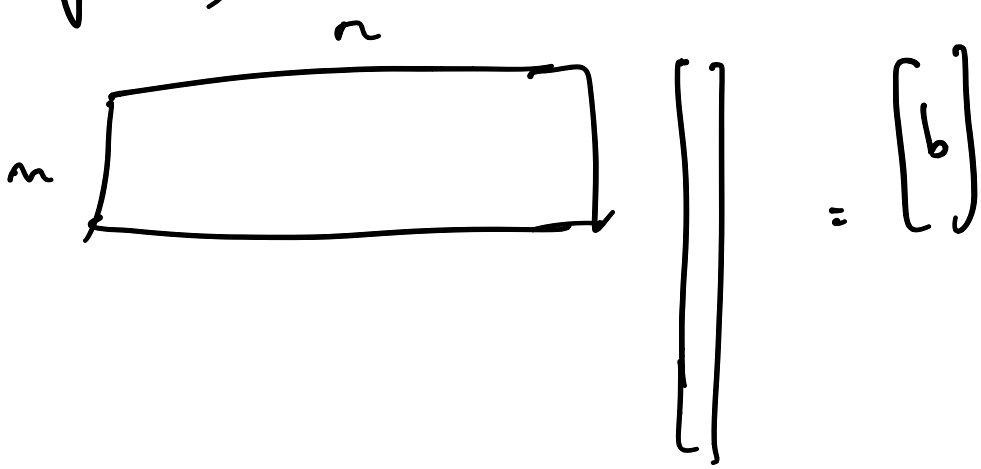
and finding unique solution to $=$ for remaining variables.

Can say more: Extreme points of $P = \{x : Ax = b, x \geq 0\}$ are the basic feasible solutions (bfs), feasible solns obtained as follows:

- Remove redundant rows from A ( )



- Choose m columns B of A, ( )



- Solve $A_B x_B = 0$, set

$$x^*_i = \begin{cases} & i \in B \\ & \text{else} \end{cases}$$

$$\boxed{\{\text{bfs}\} = \{\text{extreme pts}\}.}$$

Corollary of Faces Theorem

facets are the maximal nontrivial faces of a nonempty polyhedron P.

Pf: Exercise.

Corollary of vertex theorem

vertices are the minimal nontrivial

Facets.

Pf: